NACA

RESEARCH MEMORANDUM

DETERMINATION OF MINIMUM MOMENTS OF INERTIA OF ARBITRARILY SHAPED AREAS, SUCH AS HOLLOW TURBINE BLADES

By Sel Gendler and Donald F. Johnson

Lewis Flight Propulsion Laboratory
Cleveland, Ohio

TECHNICAL EDITING WAIVED

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

WASHINGTON
February 24, 1950

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

RESEARCH MEMORANDUM

DETERMINATION OF MINIMUM MOMENTS OF INERTIA OF ARBITRARILY SHAPED AREAS, SUCH AS HOLLOW TURBINE BLADES

By Sel Gendler and Donald F. Johnson

SUMMARY

A simple accurate method is presented for approximately determining the minimum moment of inertia of an arbitrarily shaped area, such as the section of a hollow turbine blade. The practical application of this method involves a simple routine tabular procedure. Incidental to finding the minimum moment of inertia, this tabular procedure also gives the area, the position of the center of gravity, the moment of inertia about any desired axis, the product of inertia, and the principal axes of inertia. Two examples are worked out in detail: an ellipse tilted at an angle of 30°, and an airfoil section.

In general, comparison with the known values for the ellipse showed agreement to within 0.5 percent. This method was also used to calculate the variation in minimum moment of inertia along the length of a typical hollow turbine blade.

INTRODUCTION

In several types of problems involving vibrations of an elastic body, it is necessary to know the area, the plane product and the moment of inertia about a given axis, the minimum moment of inertia, and the variations of these quantities along an axis through the body. The most obvious method, that of "counting squares", can be applied to obtain the area, the product of inertia, and the moment of inertia about a given axis. The minimum moment of inertia can be approximated by a trial-and-error procedure of passing several axes through the center of gravity and calculating the moment of inertia about each of these axes. This method is long and laborious and the accuracy attained depends a great deal upon the skill of the operator. A more expedient method, which is in general use, is that of numerical integration.

As a result of an investigation, conducted at the NACA Lewis laboratory, of the vibration problems of compressor and turbine blades in general and hollow turbine blades in particular, a simple accurate method of calculating the moment of inertia about a given axis and the minimum moment of inertia of arbitrarily shaped sections has been developed. This method combines the well-known theories of moments of inertia and numerical integration to provide a simple straightforward tabular procedure for calculating the section properties. The area, the position of the center of gravity, the moment of inertia about any given axis, and the product of inertia are obtained incidental to finding the minimum moment of inertia. The values obtained are dependable and may be calculated to any reasonable degree of accuracy without unduly increasing the amount of labor. The procedure is illustrated by two simple examples, an ellipse tilted at an angle and an airfoil section, and is finally applied to finding the variation of the minimum moment of inertia along the length of a twisted and tapered hollow turbine blade.

SYMBOLS

The following symbols are used in this report:

A	cross-sectional area
A_r	cross-sectional area of root section
C	boundary
c	chord
F	product of inertia about the x,y-axes
F_0	product of inertia with respect to axes through centroid parallel to x,y-axes
I_m	minimum moment of inertia
$I_{m,r}$	minimum moment of inertia at root
I_x	moment of inertia about x-axis
$I_{x,o}$	moment of inertia about axis through centroid parallel to x-axis
I_y	moment of inertia about y-axis

$I_{y,o}$ moment of inertia about axis through centroid parallel to y-axis

$I_m/I_{x,o}$ minimum x inertia ratio

$I_m/I_{y,o}$ minimum y inertia ratio

l length of blade

n number of stations into which area is divided

x,y reference coordinate axes

x',y' principal axes about one of which moment of inertia of area is minimum

$\bar{x},\bar{y}$ location of centroid of area referred to x,y-axes

x_0,y_0 location of origin of x',y'-axes referred to x,y-axes

α inertia parameter, $2\,F_0/(I_{y,o} - I_{x,o})$

θ angle of x'-axis to x-axis measured in positive counter-clockwise direction

METHOD

Let x,y be any set of reference axes with respect to which the coordinates of the area under consideration are known (fig. 1). Let x',y' be the principal axes at the point (x_0,y_0), such that the moment of inertia of the area is a minimum about the x'-axis, and let $\bar{x},\bar{y}$ be the coordinates of the centroid of the area with respect to the x,y axes. Then (reference 1):

$$\left.\begin{aligned} x_0 &= \bar{x} \\ y_0 &= \bar{y} \end{aligned}\right\} \tag{1}$$

$$\tan 2\theta = \frac{2\,F_0}{I_{y,o}-I_{x,o}} \tag{2}$$

Hence, it can be derived that

$$I_m = -F_0 \cot \theta + I_{y,o}$$

$$= \frac{1}{2} (I_{y,o}-I_{x,o}) \left(1 \pm \sqrt{1+\alpha^2}\right) + I_{x,o} \tag{3}$$

where

$$\alpha = \frac{2 F_0}{I_{y,o}-I_{x,o}} \tag{4}$$

Equation (1) states the well-known fact that the moment of inertia will be least about an axis passing through the centroid. Equation (2) gives the angle that this axis makes with the x-axis. In equation (3), which gives the minimum moment of inertia, the sign before the radical is so chosen that the first term on the right side of the equation is negative.

If the equations of the contour are known, equations (1) to (3) can be solved exactly for the desired quantities. (See the appendix.) These equations, however, are generally unknown and approximate methods must be used.

Assume that the values of y defining the area boundary are known for n stations along the x-axis. These values may be obtained by direct measurement from a drawing, or as is generally the case for airfoil sections, they may be directly obtained from a table that gives the values of y in percentage of chord for a number of stations along the chord. The following equations then approximately hold:

1181

$$\left.\begin{aligned}
A &= -\sum_{i=1}^{n} y_i \, \Delta x_i \\
\bar{x} &= -\frac{1}{A}\sum_{i=1}^{n} x_i y_i \, \Delta x_i \\
\bar{y} &= -\frac{1}{2A}\sum_{i=1}^{n} {y_i}^2 \, \Delta x_i \\
F &= -\frac{1}{2}\sum_{i=1}^{n} x_i {y_i}^2 \, \Delta x_i \\
I_x &= -\frac{1}{3}\sum_{i=1}^{n} {y_i}^3 \, \Delta x_i \\
I_y &= -\sum_{i=1}^{n} y_i {x_i}^2 \, \Delta x_i
\end{aligned}\right\} \tag{5}$$

Also

$$I_{x,o} = I_x - \bar{y}^2 A$$

$$I_{y,o} = I_y - \bar{x}^2 A$$

$$F_0 = F - \bar{x}\,\bar{y}\,A$$

The values x_i and y_i should be taken at the midpoint of the interval Δx_i. It is convenient to take equal intervals Δx, because by so doing, Δx_i moves to a position in front of the summation sign and thus considerably shortens computation.

The problem then reduces to one of tabulating n values of x and y and obtaining the sums of the various products indicated in equations (5). The minimum moment of inertia is then found by use of equations (3) and (4). The accuracy with which the factors in equations (5) are obtained increases with n, the number of stations used. In general, it will be seen from the examples that good accuracy can be obtained by use of relatively few stations.

In carrying out the preceding calculations, care must be taken as to the sign of the quantity Δx. The summation, as shown by the arrows in figures 1 and 2, is in such a direction that the area under consideration is always to the left. If the summation is in the direction of increasing x, the sign of Δx should be positive; if the summation is in the direction of decreasing x, the sign of Δx should be negative. Thus, in figure 1, Δx is positive for curves C_1 and C_3 and negative for curves C_2 and C_4. This convention has been followed in the derivation of equations (5) and must be followed in their application.

Graphical solution of I_m. - Equation (3) is made nondimensional if both members are divided by $I_{x,o}$.

$$\frac{I_m}{I_{x,o}} = 1 + \frac{1}{2}\left[1 \pm (1+\alpha^2)^{\frac{1}{2}}\right]\left(\frac{I_{y,o}}{I_{x,o}} - 1\right) \tag{6}$$

Equation (6) represents an infinite family of straight lines with the minimum inertia ratio, $I_m/I_{x,o}$, as ordinate and $(I_{y,o}/I_{x,o}) - 1$ as abscissa. (In actual practice, it will be found more convenient to use $I_{y,o}/I_{x,o}$ as the abscissa.) All the lines have a common intercept at $I_m/I_{x,o} = 1$ and the slopes of the lines depend on the inertia parameter α as follows:

$$\text{Slope} = \frac{1}{2}\left[1 \pm (1+\alpha^2)^{\frac{1}{2}}\right]$$

It is evident that the abscissa cannot become less than 0 and the ordinate cannot be less than 0 or greater than 1. Moreover, there will be no lines with slopes between 0 and 1. Such a family is plotted in figure 3. The lines giving the maximum value of $I_m/I_{x,o}$ will be merely the reflections of the lines in figure 3 about the point (1,1). If the values of α, $I_{x,o}$, and $I_{y,o}$ have been calculated for a section, the values of I_m can be obtained from figure 3.

Example 1. - Figure 2(a) shows an ellipse tilted at an angle of 30° to the x-axis. The equation of this ellipse is $\frac{x'^2}{9} + y'^2 = 1$. The area, the location of the centroid, the moment of inertia about the x-axis, the principal axes through the centroid, and the minimum moment of inertia are to be determined. The ellipse was divided into ten intervals. The quantities in equations (5) are calculated as shown in table I. The following values were obtained:

$$A = 9.516 \qquad I_x = 7.212$$

$$\bar{x} = 0.000 \qquad I_m = 2.363$$

$$\bar{y} = 0.000 \qquad \theta = 0.521 \text{ radian}$$

The exact values for the ellipse are:

$$A = 9.425 \qquad I_x = 7.069$$

$$\bar{x} = 0.000 \qquad I_m = 2.356$$

$$\bar{y} = 0.000 \qquad \theta = 0.524 \text{ radian}$$

The decrease in percentage error of I_m due to choosing a larger number of intervals over the ellipse is shown in the following table:

Error in calculated value of I_m, percent	Number of stations
10	2
2	4
.3	10
.1	20

Example 2. - A Clark Y airfoil is shown in figure 2(b). The shape of this airfoil in terms of the chord is given in table II. The quantities in equations (5) are obtained as shown in table III. The final answers are:

$$A = 815.7\left(\frac{c^2}{10^4}\right) \qquad I_x = 26{,}870\left(\frac{c^4}{10^8}\right)$$

$$\bar{x} = 42.05\left(\frac{c}{10^2}\right) \qquad I_y = 1{,}894{,}000\left(\frac{c^4}{10^8}\right)$$

$$\bar{y} = 4.853\left(\frac{c}{10^2}\right) \qquad I_m = 7087\left(\frac{c^4}{10^8}\right)$$

$$F = 150{,}600\left(\frac{c^4}{10^8}\right) \qquad \theta = 0.0358 \text{ radian}$$

DISCUSSION OF RESULTS

Examples 1 and 2 with tables I and III show an application of the method for obtaining the minimum moment of inertia of arbitrarily shaped areas. As has already been pointed out, equations (5) are approximate, the degree of approximation depending upon the number of stations considered. It is evident that in the case of the ellipse, the error involved in the use of ten stations is less than 0.5 percent. Thus, it is unnecessary to consider many stations to achieve good accuracy.

The number of stations required to give a certain degree of accuracy will, of course, depend on the shape of the area under consideration. In order to obtain good over-all accuracy, areas that have sharp peaks in their contours, such as a cross-shaped section, will require more stations in the vicinity of the peaks than in the more uniform parts. Fortunately, most objects dealt with in physical problems do not have such sharp peaks. If such an object is being considered, however, care must be taken in selecting the number of intervals and in distributing the intervals. It may be advantageous to take small intervals for certain parts of the contour and large intervals for the rest of the contour.

Possible sources of error should be pointed out. If the angle θ is very close to 45°, the denominator of equation (2) becomes very small. If small errors do occur in the calculation of $I_{y,o}$ and $I_{x,o}$, it is possible that the wrong sign will be obtained for the square root of equation (3). If $I_{x,o}$ is a small quantity, a large error in the value of I_m could be introduced. Such an error could easily be prevented because by inspection, it should be obvious whether θ is 45° or -45°. In any case, it is better to choose the x',y'-axes such that θ will not be close to 45°.

If the coordinate axes are so chosen that the angle θ is known to be small, an approximate form of equation (3) can be used. For small θ, α is small and α^2 becomes negligible compared to 1. Equation (3) then becomes

$$I_m = \frac{1}{2}(I_{y,o}-I_{x,o})(1 \pm 1) + I_{x,o}$$

Thus, I_m as a minimum will equal either $I_{y,o}$ or $I_{x,o}$, whichever is smaller. If, however, the absolute value of $(I_{y,o}-I_{x,o})$ is large compared to $I_{x,o}$, the neglect of even a small quantity α^2 can introduce large errors in I_m.

The method described was used to calculate the minimum moment of inertia of various sections along the length of a hollow turbine blade. Five sections, as shown in figure 4, including the tip and root section were used. The results and the area variation are plotted in dimensionless coordinates, as shown in figure 5. These variations must be known in order to calculate accurately the natural vibration frequencies of the turbine blades.

SUMMARY OF RESULTS

A simple tabular method for obtaining the minimum moment of inertia of an arbitrarily shaped section was developed. The area, the position of the centroid, the moment of inertia about an arbitrary set of axes, and the product of inertia were obtained as part of this tabular procedure. Although the accuracy obtained using a given number of stations depends upon the irregularity of the area under consideration, it was shown that, in general, high accuracy could be obtained with the use of comparatively few stations. In the case of an ellipse tilted at 30°, accuracy of greater than 0.5 percent was obtained by the use of ten stations. A final application was made in calculating the variation of the minimum moment of inertia along the length of a hollow turbine blade.

Lewis Flight Propulsion Laboratory,
National Advisory Committee for Aeronautics,
Cleveland, Ohio.

APPENDIX

EXACT SOLUTION OF MINIMUM MOMENT OF INERTIA

Minimum moment of inertia. - Let x,y be the coordinates of a point of a closed contour relative to a given set of x,y-axes; let x',y' be the coordinates of this point relative to the principal axes of the contour (about which the moment of inertia is a minimum); and let θ be the angle between the two sets of axes. Then,

$$I_m = \cot\theta\left(\bar{x}\,\bar{y}A-F\right) + I_y-\bar{x}^2A \qquad \text{(A1)}$$

where

$$\tan 2\theta = \frac{2(F-\bar{x}\,\bar{y}A)}{(I_y-\bar{x}^2A)-(I_x-\bar{y}^2A)} = \frac{2F_0}{I_{y,o}-I_{x,o}} \qquad \text{(A2)}$$

By taking $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$ and letting $\alpha = \dfrac{2F_0}{I_{y,o}-I_{x,o}}$, the following equation is obtained by substituting in equation (A1) and carrying through the proper trigonometric manipulation:

$$I_m = \frac{1}{2}\left(I_{y,o}-I_{x,o}\right)\left[1 \pm \left(1+\alpha^2\right)^{\frac{1}{2}}\right] + I_{x,o} \qquad \text{(A3)}$$

The sign preceding the radical is determined by the condition that $I_{x,o}-I_m$ and $I_{y,o}-I_m$ must each exceed zero. Thus, if $(I_{y,o}-I_{x,o}) > 0$, a minus sign is used and if $(I_{x,o}-I_{y,o}) > 0$, a plus sign is used.

Exact solution. - If $P(x,y)$ and $\dfrac{\partial P(x,y)}{\partial y}$ are continuous within and on the boundary of region A, then by use of Gauss' theorem (reference 2)

$$\iint_A \frac{\partial P(x,y)}{\partial y}\,dA = -\int_C P(x,y)\,dx \qquad \text{(A4)}$$

where the integration on the right is performed in a positive sense around the boundary C.

From equation (A4) and the fact that the line integral around a closed boundary $\int_C F(x)\,dx$ equals zero, the following equations are evident:

$$\left.\begin{aligned}
A &= \iint_A dA = -\int_C y\,dx \\
\bar{x} &= \frac{1}{A}\iint_A x\,dA = -\frac{1}{A}\int_C xy\,dx \\
\bar{y} &= \frac{1}{A}\iint_A y\,dA = -\frac{1}{2A}\int_C y^2\,dx \\
F &= \iint_A xy\,dA = -\frac{1}{2}\int_C xy^2\,dx \\
I_x &= \iint_A y^2\,dA = -\frac{1}{3}\int_C y^3\,dx \\
I_y &= \iint_A x^2\,dA = -\int_C x^2 y\,dx
\end{aligned}\right\} \tag{A5}$$

Also

$$I_{x,o} = I_x - \bar{y}^2 A$$

$$I_{y,o} = I_y - \bar{x}^2 A$$

$$F_0 = F - \bar{x}\,\bar{y}A$$

If the equation of the boundary of the section under consideration is known, the quantities in equation (A5) can then be exactly determined by carrying out the line integration indicated. The integration is in a direction such that the area is always to the left.

REFERENCES

1. Timoshenko, S.: Strength of Materials, pt. I. D. Van Nostrand, Inc. (New York), 2d ed., 1940, pp. 348-353.

2. Osgood, William F.: Advanced Calculus. The Macmillan Co. (New York), 1925, pp. 222-224.

1181

TABLE I - SOLUTION OF EQUATIONS (5) FOR ELLIPSE

Δx	x	y	y^2	y^3	xy^2	x^2y	xy
-0.52915	2.3812	-0.6842	0.4681				
	1.8520	-.1067	.0114				
	1.3229	.3273	.1071				
	.7937	.6889	.4746				
	.2646	.9973	.9946				
	-.2646	1.2592	1.5856				
	-.7937	1.4745	2.1742				
	-1.3229	1.6367	2.6788				
	-1.8520	1.7263	2.9801				
	-2.3812	1.6727	2.7979				
Total		8.9920	14.2724	20.4439	-15.9532	16.1181	-11.4320
0.52915	2.3812	-1.6727	2.7979				
	1.8520	-1.7263	2.9801				
	1.3229	-1.6367	2.6788				
	.7937	-1.4745	2.1742				
	.2646	-1.2592	1.5856				
	-.2646	-.9973	.9946				
	-.7937	-.6889	.4746				
	-1.3229	-.3273	.1071				
	-1.8520	.1067	.0114				
	-2.3812	.6842	.4681				
Total		-8.9920	14.2724	-20.4439	15.9532	-16.1181	-11.4320

NACA

$$A = -\Sigma y \Delta x = 9.5162$$

$$\bar{x} = -\frac{\Sigma xy \Delta x}{A} = 0.0000$$

$$\bar{y} = -\frac{\Sigma y^2 \Delta x}{2A} = 0.0000$$

$$F = -\frac{\Sigma xy^2 \Delta x}{2} = 8.4416$$

$$I_x = -\frac{\Sigma y^3 \Delta x}{3} = 7.2119$$

$$I_y = -\Sigma x^2 y \Delta x = 17.0578$$

$$I_{x,o} = I_x - \bar{y}^2 A = 7.2119$$

$$I_{y,o} = I_y - \bar{x}^2 A = 17.0578$$

$$F_O = F - \bar{x}\bar{y} A = 8.4416$$

$$\alpha = \frac{2F_O}{I_{y,o} - I_{x,o}} = 1.7147$$

$$I_m = \frac{I_{y,o} - I_{x,o}}{2}\left[1-(1+\alpha^2)^{\frac{1}{2}}\right] + I_{x,o} = 2.3628$$

$$\theta = \frac{1}{2}\tan^{-1}\frac{2F_O}{I_{y,o} - I_{x,o}} = 0.5214 \text{ radian}$$

TABLE II - SHAPE OF CLARK Y AIRFOIL

1181

Percentage of chord		
Distance from leading edge	Upper camber	Lower camber
0	3.50	3.50
1.25	5.45	1.93
2.50	6.50	1.47
5.00	7.90	.93
7.50	8.85	.63
10.00	9.60	.42
15.00	10.69	.15
20.00	11.36	.03
30.00	11.70	0
40.00	11.40	0
50.00	10.52	0
60.00	9.15	0
70.00	7.35	0
80.00	5.22	0
90.00	2.80	0
95.00	1.49	0
100.00	.12	0

TABLE III - SOLUTION OF EQUATIONS (5) FOR CLARK Y AIRFOIL

Δx	x	y	y^2	y^3	xy^2	x^2y	xy
-10	10	9.60	92.1600				
	20	11.36	129.0496				
	30	11.70	136.8900				
	40	11.40	129.9600				
	50	10.52	110.6704				
	60	9.15	83.7225				
	70	7.35	54.0225				
	80	5.22	27.2484				
	90	2.80	7.8400				
Total		79.10	771.5634	7925.4640	30,031.6090	185,617.0000	3389.3000
-5	2.5	6.500	42.2500				
	97.5	.805	.6480				
Total		7.305	42.8980	275.1466	168.8050	7693.1563	94.7375
10	10	0.42	.1764				
	20	.03	.0009				
	30	0	0				
	40	0	0				
	50	0	0				
	60	0	0				
	70	0	0				
	80	0	0				
	90	0	0				
Total		.45	.1773	0.0741	1.7820	54.0000	4.8000
5	2.5	1.47	2.1609				
	97.5	0	0				
Total		1.47	2.1609	3.1765	5.4022	9.1875	3.6750

NACA

$$A = -\Sigma y\,\Delta x = 815.7\left(\frac{c^2}{10^4}\right)$$

$$\bar{x} = -\frac{\Sigma xy\,\Delta x}{A} = 42.05\left(\frac{c}{10^2}\right)$$

$$\bar{y} = -\frac{\Sigma y^2\,\Delta x}{2A} = 4.853\left(\frac{c}{10^2}\right)$$

$$F = -\frac{\Sigma xy^2\,\Delta x}{2} = 150,600\left(\frac{c^4}{10^8}\right)$$

$$I_x = -\frac{\Sigma y^3\,\Delta x}{3} = 26,870\left(\frac{c^4}{10^8}\right)$$

$$I_y = -\Sigma x^2 y\,\Delta x = 1,894,000\left(\frac{c^4}{10^8}\right)$$

$$\alpha = 0.07169$$

$$I_m = 7087\left(\frac{c^4}{10^8}\right)$$

$$\theta = 0.0358 \text{ radian}$$

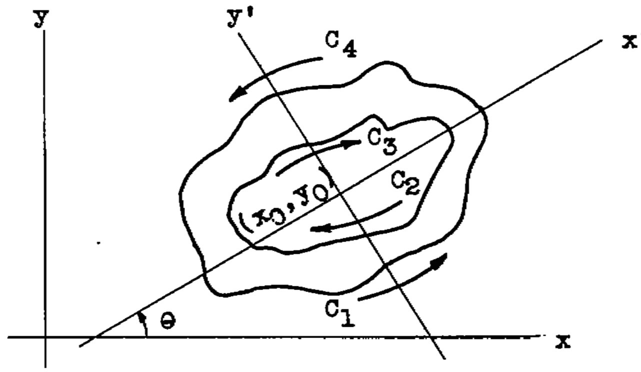
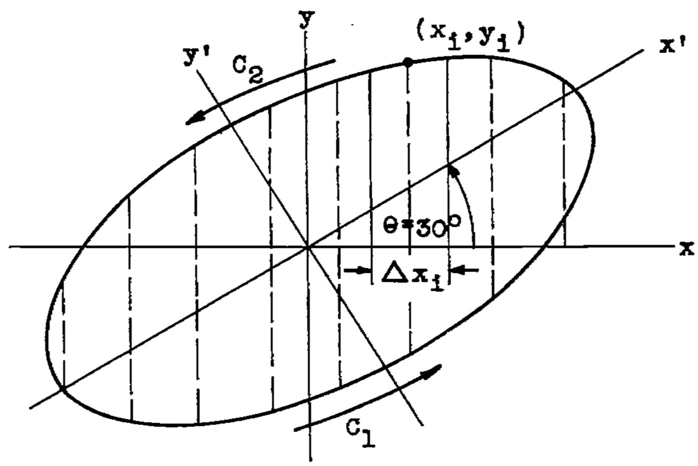
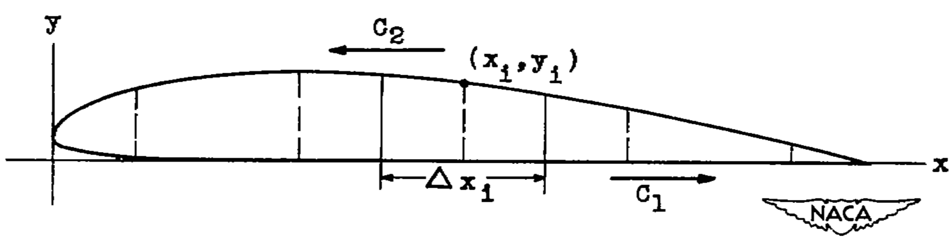


Figure 1. - Arbitrarily shaped section.



(a) Ellipse tilted at 30°.



(b) Clark Y airfoil.

Figure 2. - Areas under consideration for minimum moment of inertia.

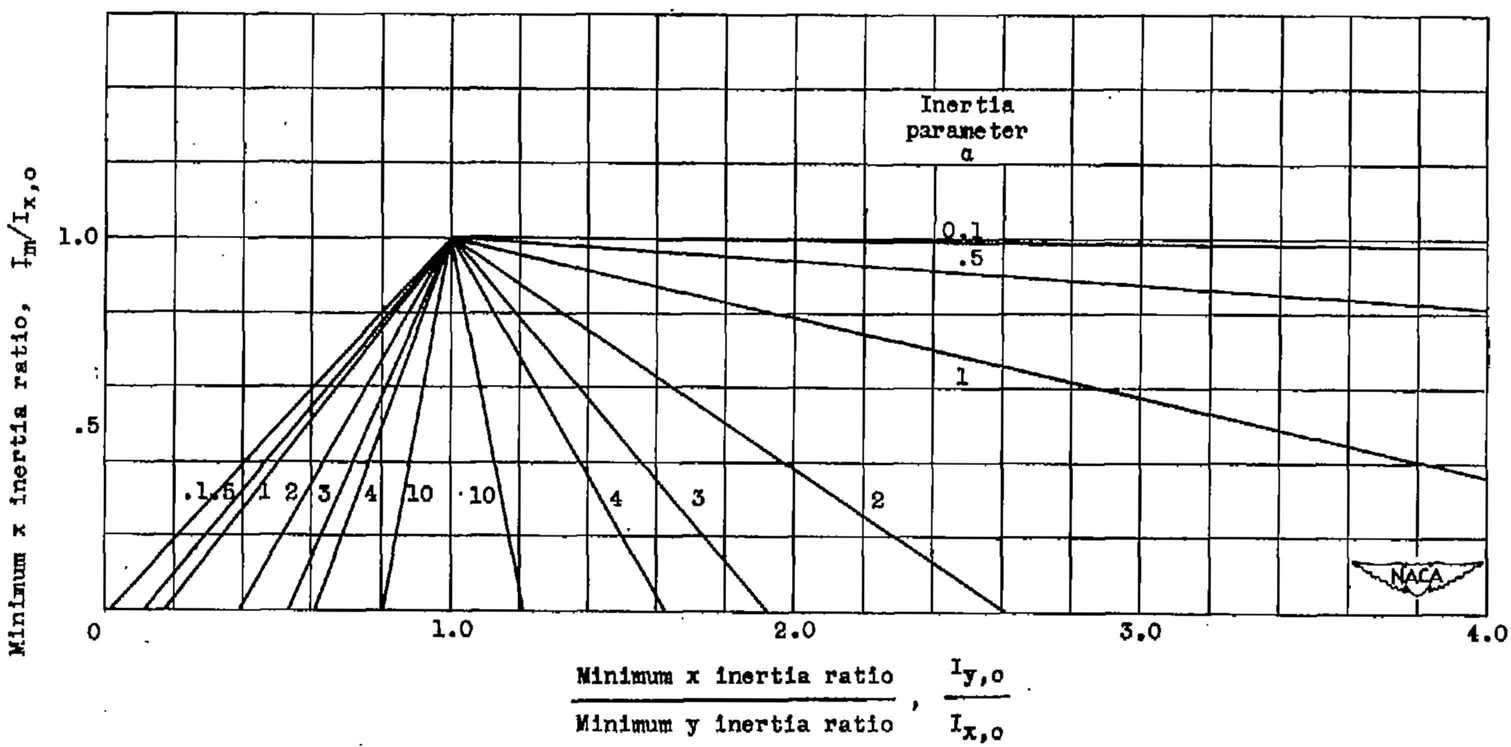


Figure 3. - Variation of minimum x inertia ratio $I_m/I_{x,o}$ with ratio of minimum x inertia ratio $I_m/x,o$ to minimum y inertia ratio $I_m/I_{y,o}$, $I_{y,o}/I_{x,o}$.

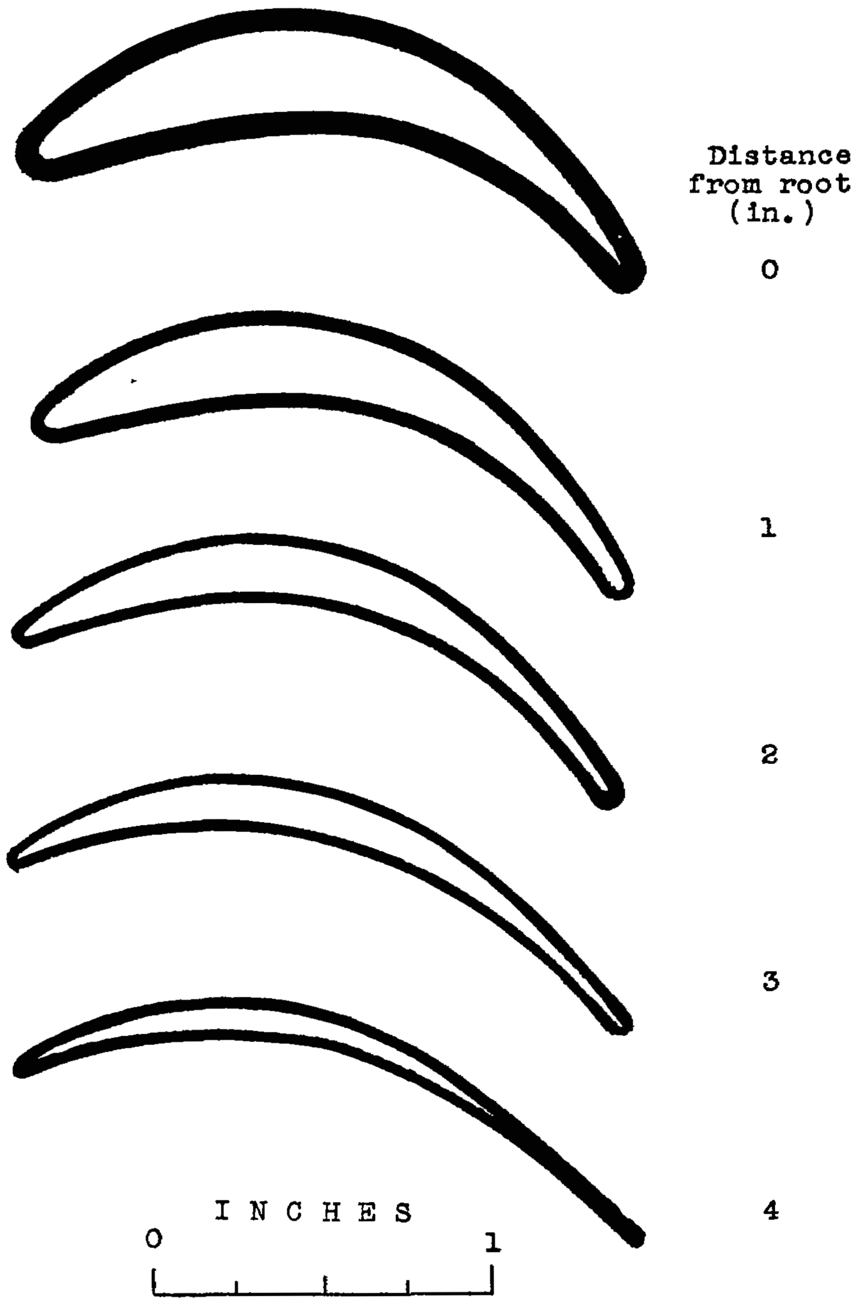


Figure 4. - Variation of cross section of hollow turbine blade. Length, 4 inches.

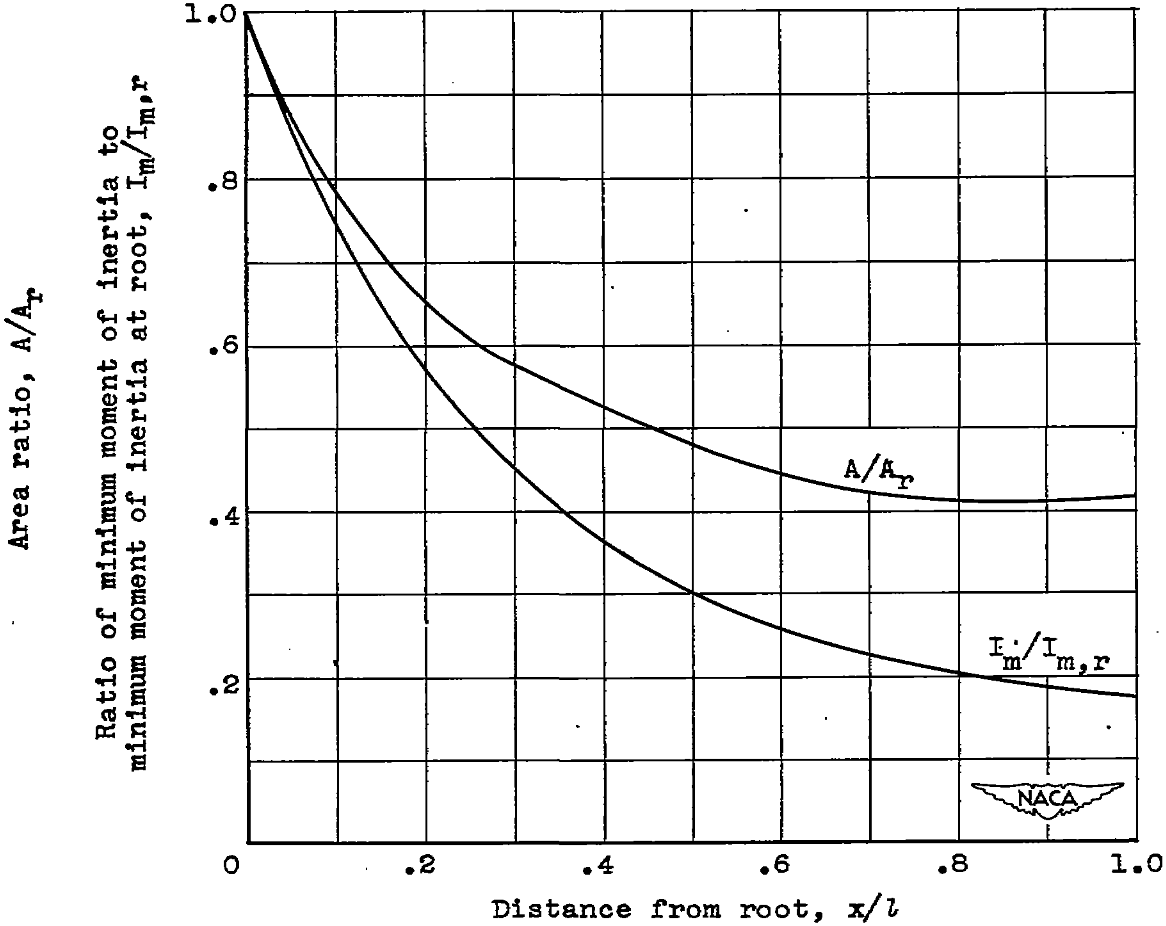


Figure 5. - Variation of minimum moment of inertia and area along length of typical hollow turbine blade.